S.T.E.M. Online Advanced Readiness Diagnostic

Intake exam packet with candidate copy, full worked solutions, and grading sheet

Purpose	Determine whether a student is prepared for rigorous coaching in Calculus, Linear Algebra, ODE, Mechanics, and mathematical-physics pathways.
Time	110 minutes
Format	16 required questions across four sections, plus 2 bonus questions.
Scoring	100 marks total + 10 bonus marks. Use the grading sheet at the end for course-fit decisions.

Candidate instructions

- Show all major steps. Unsupported final answers receive little or no credit.
- A correct idea with incomplete algebra may earn partial credit; unexplained jumps generally do not.
- Use exact values whenever possible. Numerical approximations must be clearly labeled.
- For proof questions, examples do not count as proofs unless the prompt asks for a counterexample.
- For mechanics and E&M, model first: diagrams, force balances, conservation laws, or field reasoning should appear before computation.

Section weights

Section	Topic	Marks
A	High School Calculus	30
B	Basic Proof Techniques	20
C	Elementary Linear Algebra	25
D	High School Mechanics and E&M	25

Candidate Exam

Section A. High School Calculus

A1. Limit with structure, not pattern-matching *[6 marks]*

Evaluate lim(x -> 0) [sqrt(1 + 3x) - 1 - (3/2)x] / x^2.

A2. Derivative from first principles plus interpretation *[8 marks]*

Let f(x) = 1 / (x + 1).

1. Compute f'(a) from the limit definition.
2. Find the equation of the tangent line at x = 1.
3. Determine whether the tangent line lies above or below the curve near x = 1, and justify.

A3. Optimization with modeling *[8 marks]*

A rectangle has its base on the x-axis and its top two vertices on the parabola y = 12 - x^2.

Find the dimensions of the rectangle of maximum area.

A4. Integral and interpretation *[8 marks]*

Let v(t) = t^2 - 4t + 3 for 0 <= t <= 5.

1. Find the displacement on [0,5].
2. Find the total distance traveled on [0,5].
3. Explain why the two answers differ.

Section B. Basic Proof Techniques

B1. Direct proof / contradiction hybrid *[6 marks]*

Prove that if n^2 is even, then n is even.

B2. Divisibility proof *[6 marks]*

Prove that for every integer n, the quantity n^3 - n is divisible by 6.

B3. Sets and logic *[4 marks]*

Let A, B, C be sets. Prove or disprove:

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

If true, prove it by double inclusion.

B4. Counterexample detection *[4 marks]*

Determine whether each statement is true or false.

1. If ab = 0, then a = 0 or b = 0, for all real numbers a,b.
2. If AB = 0, then A = 0 or B = 0, for all 2 x 2 matrices A,B.

Give a proof for the real-number statement and a counterexample or proof for the matrix statement.

Section C. Elementary Linear Algebra

C1. System solving with interpretation *[6 marks]*

Solve the system:

$x + 2y - z = 3$

$2x + 5y - 3z = 7$

$x + 3y - 2z = 4$

Then state whether the solution is unique, and explain how you know.

C2. Linear dependence / span *[7 marks]*

Consider v1 = (1,2,1), v2 = (2,1,3), v3 = (3,3,4) in R^3.

1. Determine whether {v1, v2, v3} is linearly independent.
2. If dependent, express one vector as a linear combination of the others.
3. Geometrically interpret the result.

C3. Matrix transformation *[6 marks]*

Let A = [[2,1],[1,2]].

1. Compute A(x,y)^T.
2. Find all vectors (x,y)^T such that A(x,y)^T = (3,3)^T.
3. Explain geometrically whether this transformation can collapse the plane to a line.

C4. Eigenvalue readiness lite *[6 marks]*

Let B = [[0,1],[1,0]].

1. Find all eigenvalues of B.
2. Find a corresponding eigenvector for each.
3. Explain geometrically what B does to vectors in the plane.

Section D. High School Mechanics and E&M

D1. Newtonian modeling on an incline *[7 marks]*

A block of mass m slides down a rough incline of angle theta with coefficient of kinetic friction mu_k.

1. Draw the free-body diagram.
2. Derive the acceleration.
3. State the condition under which the block will actually slide downward.

D2. Work-energy with variable interpretation *[6 marks]*

A particle of mass m is launched straight upward from ground level with speed v0, neglecting air resistance.

1. Find the maximum height using energy.
2. Derive the same result using kinematics.
3. Explain why the two methods agree.

D3. Momentum and conservation judgment *[6 marks]*

Two objects of masses m and 2m move along a line with velocities u and -u/2, respectively. They collide and stick together.

1. Find the final velocity.
2. Determine whether kinetic energy is conserved.
3. Quantify the loss in kinetic energy.

D4. Basic E&M field-force reasoning *[6 marks]*

Two point charges, +q and +4q, are fixed a distance d apart.

1. Find the point on the line joining them where the net electric field is zero.
2. Can the net electric potential be zero at a point between them? Justify.
3. A test charge +Q is placed at the zero-field point. What is the net force on it?

Bonus questions (optional)

X1. Cross-topic maturity question *[5 marks]*

A student claims: If a function has derivative zero at a point, then that point must be a local maximum or minimum.

Is this statement true or false? Give a proof or counterexample.

X2. Mathematical physics bridge question *[5 marks]*

Suppose vectors u,v in R^2 satisfy u · v = 0. Does this imply one of the vectors must be zero?

Prove or disprove, and compare this with the statement If ab = 0 then a = 0 or b = 0 for real numbers.

Worked Solutions Key

Section A solutions

A1. Limit with structure, not pattern-matching [6 marks]

Let $N(x) = \sqrt{1 + 3x} - 1 - (3/2)x$. Rationalize the first difference: $\sqrt{1 + 3x} - 1 = 3x / (\sqrt{1 + 3x} + 1)$.

Then $N(x) = 3x/(\sqrt{1 + 3x} + 1) - (3/2)x = 3x[1/(\sqrt{1 + 3x} + 1) - 1/2]$.

Combine terms: $N(x) = 3x[1 - \sqrt{1 + 3x}] / [2(\sqrt{1 + 3x} + 1)]$.

Rationalize again: $1 - \sqrt{1 + 3x} = -3x / (1 + \sqrt{1 + 3x})$.

Hence $N(x) = -9x^2 / [2(1 + \sqrt{1 + 3x})^2]$.

Divide by x^2 and take $x \to 0$: limit $= -9 / [2(2)^2] = -9/8$.

A2. Derivative from first principles plus interpretation [8 marks]

$f'(a) = \lim_{h\to 0} [1/(a+h+1) - 1/(a+1)] / h$.

Combine fractions: numerator $= [(a+1) - (a+h+1)] / [(a+h+1)(a+1)] = -h / [(a+h+1)(a+1)]$.

Thus $f'(a) = \lim_{h\to 0} -1 / [(a+h+1)(a+1)] = -1 / (a+1)^2$.

At $x = 1$, point on curve is $(1,1/2)$ and slope is $-1/4$. Tangent line: $y - 1/2 = (-1/4)(x - 1)$.

Equivalently $y = 3/4 - x/4$.

Since $f''(x) = 2/(x+1)^3 > 0$ near $x = 1$, the curve is concave up there, so the tangent line lies below the curve locally.

A3. Optimization with modeling [8 marks]

If the upper corners are at $(x, 12 - x^2)$ and $(-x, 12 - x^2)$, then width $= 2x$ and height $= 12 - x^2$ with $0 \le x \le \sqrt{12}$.

Area $A(x) = 2x(12 - x^2) = 24x - 2x^3$.

$A'(x) = 24 - 6x^2 = 6(4 - x^2)$, so critical point is $x = 2$ in the domain.

$A''(x) = -12x$, hence $A''(2) = -24 < 0$, so $x = 2$ gives a maximum.

Width = 4, height = 12 - 4 = 8. The maximal rectangle has dimensions 4 by 8.

A4. Integral and interpretation [8 marks]

Displacement = integral from 0 to 5 of $(t^2 - 4t + 3)\,dt = [t^3/3 - 2t^2 + 3t]_0^5 = 125/3 - 50 + 15 = 20/3$.

Factor $v(t) = (t-1)(t-3)$, so the sign changes at $t = 1$ and $t = 3$.

Total distance $= \int_0^1 v(t)\,dt - \int_1^3 v(t)\,dt + \int_3^5 v(t)\,dt$.

Using $F(t) = t^3/3 - 2t^2 + 3t$, these pieces are $F(1)-F(0)=4/3$, $-(F(3)-F(1)) = 4/3$, and $F(5)-F(3)=20/3$.

Total distance $= 4/3 + 4/3 + 20/3 = 28/3$.

They differ because displacement is signed change in position, while total distance counts all motion as positive.

Section B solutions

B1. Direct proof / contradiction hybrid [6 marks]

Prove the contrapositive: if n is odd, then n^2 is odd.

Let $n = 2k + 1$ for some integer k. Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd.

Therefore the contrapositive is true, so if n^2 is even, then n is even.

B2. Divisibility proof [6 marks]

Factor: $n^3 - n = n(n^2 - 1) = n(n-1)(n+1)$.

These are three consecutive integers. Among any three consecutive integers, one is divisible by 3 and at least one is even.

Therefore the product is divisible by both 2 and 3, hence divisible by 6.

B3. Sets and logic [4 marks]

The statement is true.

If x is in $A \cap (B \cup C)$, then x is in A and x is in B or C. Thus x is in $A \cap B$ or x is in $A \cap C$, so x is in $(A \cap B) \cup (A \cap C)$.

Conversely, if x is in $(A \cap B) \cup (A \cap C)$, then either x is in $A \cap B$ or in $A \cap C$. In either case x is in A and x is in $B \cup C$. Hence x is in $A \cap (B \cup C)$.

So the two sets are equal by double inclusion.

B4. Counterexample detection [4 marks]

For real numbers, the statement is true: if ab = 0 and a is nonzero, then dividing by a gives b = 0. Symmetrically, if b is nonzero then a = 0.

For matrices, the statement is false. Take A = [[1,0],[0,0]] and B = [[0,0],[1,0]].

Then AB = [[0,0],[0,0]] although A is not the zero matrix and B is not the zero matrix.

So the zero-product property holds in R but fails for 2 x 2 matrices.

Section C solutions

C1. System solving with interpretation [6 marks]

Subtract the first equation from the third: $y - z = 1$.

Subtract twice the first equation from the second: $y - z = 1$ again.

Thus only two independent equations remain. From $y = z + 1$, the first equation gives $x + 2(z + 1) - z = 3$, so $x + z = 1$.

Hence $x = 1 - z$ and $y = z + 1$. Let $z = t$. Then $(x,y,z) = (1 - t, 1 + t, t)$.

There is not a unique solution; there are infinitely many solutions forming a line in R^3 because one free parameter remains.

C2. Linear dependence / span [7 marks]

Observe $v1 + v2 = (1,2,1) + (2,1,3) = (3,3,4) = v3$.

Therefore the set is linearly dependent.

One relation is $v3 = v1 + v2$, or equivalently $v1 + v2 - v3 = 0$.

Geometrically, all three vectors lie in the span of v1 and v2; v3 adds no new direction.

C3. Matrix transformation [6 marks]

$A(x,y)^T = (2x + y, x + 2y)^T$.

Solve $2x + y = 3$ and $x + 2y = 3$. Subtracting gives $x - y = 0$, so $x = y$. Then $3x = 3$, hence $x = y = 1$.

There is exactly one solution vector: $(1,1)^T$.

The determinant of A is $4 - 1 = 3$, which is nonzero. Therefore the map is invertible and cannot collapse the plane to a line.

C4. Eigenvalue readiness lite [6 marks]

The characteristic polynomial is $\det(B - \lambda I) = \lambda^2 - 1$, so the eigenvalues are $\lambda = 1$ and $\lambda = -1$.

For $\lambda = 1$, solve $Bv = v$. This gives $y = x$, so one eigenvector is $(1,1)$.

For $\lambda = -1$, solve $Bv = -v$. This gives $y = -x$, so one eigenvector is $(1,-1)$.

Geometrically, B swaps the two coordinates: (x,y) maps to (y,x). It is reflection across the line $y = x$.

Section D solutions

D1. Newtonian modeling on an incline [7 marks]

Forces: weight mg downward, normal force N perpendicular to the plane, and kinetic friction $f_k = \mu_k N$ up the plane since motion is down the plane.

Resolve along the plane: component of gravity down the plane is $mg \sin(\theta)$. Perpendicular to plane: $N = mg \cos(\theta)$.

Net force down the plane is $mg \sin(\theta) - \mu_k mg \cos(\theta)$.

Therefore $a = g[\sin(\theta) - \mu_k \cos(\theta)]$.

For the block to slide downward, we need $\sin(\theta) > \mu_k \cos(\theta)$, equivalently $\tan(\theta) > \mu_k$.

D2. Work-energy with variable interpretation [6 marks]

At maximum height the speed is zero. Conservation of mechanical energy gives $(1/2)mv_0^2 = mgh_{max}$, so $h_{max} = v_0^2 / (2g)$.

Using kinematics with $v^2 = v_0^2 + 2a(y - y_0)$, and setting $v = 0$ and $a = -g$, we get $0 = v_0^2 - 2gh_{max}$.

Hence $h_{max} = v_0^2 / (2g)$ again.

They agree because both are valid consequences of the same constant-gravity model without air resistance.

D3. Momentum and conservation judgment [6 marks]

Initial momentum $= m(u) + 2m(-u/2) = mu - mu = 0$.

Since the objects stick together, the total mass is $3m$ and final momentum is $3m v_f$. Therefore $v_f = 0$.

Initial kinetic energy $= (1/2)mu^2 + (1/2)(2m)(u/2)^2 = (1/2)mu^2 + mu^2/4 = 3mu^2/4$.

Final kinetic energy $= 0$ because $v_f = 0$.

So kinetic energy is not conserved; the loss is $3mu^2/4$.

D4. Basic E&M field-force reasoning [6 marks]

Place $+q$ at $x = 0$ and $+4q$ at $x = d$. Between them the fields oppose, so set magnitudes equal: $kq/x^2 = 4kq/(d - x)^2$.

Thus $(d - x)^2 = 4x^2$, and because $0 < x < d$ we take $d - x = 2x$, giving $x = d/3$.

So the zero-field point lies between the charges, one-third of the way from $+q$ toward $+4q$.

The electric potential cannot be zero between them because $V = kq/x + 4kq/(d - x)$, a sum of positive terms.

At the zero-field point the net electric field is zero, so a test charge $+Q$ experiences net force $F = QE = 0$.

Bonus solutions

X1. Cross-topic maturity question [5 marks]

The statement is false. Counterexample: $f(x) = x^3$.

Then $f'(x) = 3x^2$, so $f'(0) = 0$, but $x = 0$ is neither a local maximum nor a local minimum because f changes from negative to positive values through 0.

A zero derivative is necessary for a smooth interior extremum, but not sufficient.

X2. Mathematical physics bridge question [5 marks]

The statement is false. In R^2, nonzero perpendicular vectors can have dot product zero, for example $u = (1,0)$ and $v = (0,1)$.

So $u \cdot v = 0$ does not imply one vector is zero.

This differs from real numbers, where $ab = 0$ implies $a = 0$ or $b = 0$ because R has the zero-product property.

The dot product is not ordinary multiplication; it combines both vectors and can vanish because of orthogonality rather than because one input is zero.

Grading Sheet and Course-Fit Rubric

Question-level rubric

Question	Max	Core scoring criteria	Common red flags
A1	6	Correct algebraic strategy, clean rationalization, correct limit.	Cancelling before justified; unsupported guessing.
A2	8	Limit definition, tangent line, concavity interpretation.	Quotient simplification errors; wrong tangent data.
A3	8	Model area correctly, optimize, verify maximum.	Missing domain check; derivative mistakes.
A4	8	Separate displacement from total distance with sign analysis.	Ignoring sign changes at $t = 1$ and $t = 3$.
B1-B4	20	Logical structure, proof validity, correct counterexample use.	Examples used as proofs; scalar and matrix logic conflated.
C1-C4	25	Accurate solving plus interpretation of structure and transformations.	Pure computation with no interpretation; determinant misuse.

Question	Max	Core scoring criteria	Common red flags
D1-D4	25	Model-first physics, conservation conditions, force/field clarity.	Formula dumping; sign errors; force-field-potential confusion.

Readiness bands

Band	Score range	Interpretation
Fully ready	85-100	Ready for rigorous coaching in most offerings; strong candidate for advanced pathways if proof and linear algebra are also strong.
Ready with controlled weaknesses	70-84	Can begin targeted course work, but should be streamed according to section profile.
Borderline	55-69	Needs remediation or a bridge program before full proof-heavy or advanced work.
Not ready	40-54	Foundation repair required before advanced tutoring is likely to be efficient.
Rebuild required	Below 40	Substantial prerequisite reconstruction needed.

Course-specific cutoffs

Course	Recommended threshold	Section conditions
Calculus I & II Mastery	Overall >= 65	Section A >= 18/30 and no catastrophic algebraic weakness.
Linear Algebra Mastery	Overall >= 68	Sections B + C >= 30/45.
Ordinary Differential Equations	Overall >= 72	A >= 20/30, C >= 14/25, D >= 14/25.
Physics Problem Solving: Mechanics	Overall >= 68	D >= 17/25 and A >= 16/30.
Quantum Mechanics:	Overall >= 80	A >= 22/30, B >= 14/20, C >=

Course	Recommended threshold	Section conditions
Mathematical Foundations		18/25; X2 strongly preferred.

Suggested use

- Give the written exam first under timed conditions.
- Then conduct a 10-15 minute oral review of one proof, one linear-algebra interpretation, and one mechanics-modeling solution.
- Use the oral review to distinguish memorized procedures from genuine mathematical maturity.