

Advanced Readiness Diagnostic

An intake examination for rigorous coaching in Calculus, Linear Algebra, Ordinary Differential Equations, Mechanics, and mathematical physics.

PURPOSE	Determine whether a student is prepared for rigorous coaching and advanced mathematical pathways.
TIME	110 minutes
FORMAT	16 required questions across four sections, plus 2 optional bonus questions.
SCORING	100 marks total + 10 bonus marks. Use the grading sheet for course-fit decisions.

Candidate instructions

- Show all major steps. Unsupported final answers receive little or no credit.
- A correct idea with incomplete algebra may earn partial credit; unexplained jumps generally do not.
- Use exact values whenever possible. Numerical approximations must be clearly labeled.
- For proof questions, examples do not count as proofs unless the prompt asks for a counterexample.
- For mechanics and E&M, model first: diagrams, force balances, conservation laws, or field reasoning should appear before computation.

Section weights

Section	Topic	Marks
A	High School Calculus	30
B	Basic Proof Techniques	20
C	Elementary Linear Algebra	25
D	High School Mechanics and E&M	25

CANDIDATE EXAM | SECTION A

High School Calculus

30 marks total

A1. Limit with structure, not pattern-matching**6 marks**

Evaluate the following limit:

$$\lim_{x \rightarrow 0} [\sqrt{1 + 3x} - 1 - (3/2)x] / x^2.$$

A2. Derivative from first principles plus interpretation**8 marks**Let $f(x) = 1/(x + 1)$.

- 1 Compute $f'(a)$ from the limit definition.
- 2 Find the equation of the tangent line at $x = 1$.
- 3 Determine whether the tangent line lies above or below the curve near $x = 1$, and justify.

A3. Optimization with modeling**8 marks**

A rectangle has its base on the x -axis and its top two vertices on the parabola $y = 12 - x^2$. Find the dimensions of the rectangle of maximum area.

A4. Integral and interpretation**8 marks**Let $v(t) = t^2 - 4t + 3$ for $0 \leq t \leq 5$.

- 1 Find the displacement on $[0, 5]$.
- 2 Find the total distance traveled on $[0, 5]$.
- 3 Explain why the two answers differ.

CANDIDATE EXAM | SECTION B

Basic Proof Techniques

20 marks total

B1. Direct proof / contradiction hybrid**6 marks**

Prove that if n^2 is even, then n is even.

B2. Divisibility proof**6 marks**

Prove that for every integer n , the quantity $n^3 - n$ is divisible by 6.

B3. Sets and logic**4 marks**

Let A, B, C be sets. Prove or disprove:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

If true, prove it by double inclusion.

B4. Counterexample detection**4 marks**

Determine whether each statement is true or false.

- 1** If $ab = 0$, then $a = 0$ or $b = 0$, for all real numbers a, b .
- 2** If $AB = 0$, then $A = 0$ or $B = 0$, for all 2×2 matrices A, B .

Give a proof for the real-number statement and a counterexample or proof for the matrix statement.

CANDIDATE EXAM | SECTION C

Elementary Linear Algebra

25 marks total

C1. System solving with interpretation**6 marks**

Solve the system:

$$\begin{aligned}x + 2y - z &= 3 \\2x + 5y - 3z &= 7 \\x + 3y - 2z &= 4\end{aligned}$$

Then state whether the solution is unique, and explain how you know.

C2. Linear dependence / span**7 marks**Consider $v_1 = (1,2,1)$, $v_2 = (2,1,3)$, $v_3 = (3,3,4)$ in $\mathbb{R}^3$.

- 1 Determine whether $\{v_1, v_2, v_3\}$ is linearly independent.
- 2 If dependent, express one vector as a linear combination of the others.
- 3 Geometrically interpret the result.

C3. Matrix transformation**6 marks**Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

- 1 Compute $A(x,y)^T$.
- 2 Find all vectors $(x,y)^T$ such that $A(x,y)^T = (3,3)^T$.
- 3 Explain geometrically whether this transformation can collapse the plane to a line.

C4. Eigenvalue readiness lite**6 marks**Let $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

- 1 Find all eigenvalues of B .
- 2 Find a corresponding eigenvector for each.
- 3 Explain geometrically what B does to vectors in the plane.

CANDIDATE EXAM | SECTION D

High School Mechanics and E&M

25 marks total

D1. Newtonian modeling on an incline**7 marks**

A block of mass m slides down a rough incline of angle θ with coefficient of kinetic friction μ_k .

- 1 Draw the free-body diagram.
- 2 Derive the acceleration.
- 3 State the condition under which the block will actually slide downward.

D2. Work-energy with variable interpretation**6 marks**

A particle of mass m is launched straight upward from ground level with speed v_0 , neglecting air resistance.

- 1 Find the maximum height using energy.
- 2 Derive the same result using kinematics.
- 3 Explain why the two methods agree.

D3. Momentum and conservation judgment**6 marks**

Two objects of masses m and $2m$ move along a line with velocities u and $-u/2$, respectively. They collide and stick together.

- 1 Find the final velocity.
- 2 Determine whether kinetic energy is conserved.
- 3 Quantify the loss in kinetic energy.

D4. Basic E&M field-force reasoning**6 marks**

Two point charges, $+q$ and $+4q$, are fixed a distance d apart.

- 1 Find the point on the line joining them where the net electric field is zero.
- 2 Can the net electric potential be zero at a point between them? Justify.
- 3 A test charge $+Q$ is placed at the zero-field point. What is the net force on it?

CANDIDATE EXAM | OPTIONAL

Bonus Questions

10 optional marks total

X1. Cross-topic maturity question**5 marks**

A student claims: If a function has derivative zero at a point, then that point must be a local maximum or minimum.

Is this statement true or false? Give a proof or counterexample.

X2. Mathematical physics bridge question**5 marks**

Suppose vectors u, v in $\mathbb{R}^2$ satisfy $u \cdot v = 0$. Does this imply one of the vectors must be zero? Prove or disprove, and compare this with the statement If $ab = 0$ then $a = 0$ or $b = 0$ for real numbers.

End of candidate exam

WORKED SOLUTIONS KEY | SECTION A

High School Calculus

Use this section after the timed diagnostic.

A1. Limit with structure, not pattern-matching

6 marks

Let $N(x) = \sqrt{1 + 3x} - 1 - (3/2)x$. Rationalize the first difference:

$$\sqrt{1 + 3x} - 1 = 3x / [\sqrt{1 + 3x} + 1].$$

Then $N(x) = 3x/[\sqrt{1 + 3x} + 1] - (3/2)x = 3x\{1/[\sqrt{1 + 3x} + 1] - 1/2\}$.

Combining terms and rationalizing again gives:

$$N(x) = -9x^2 / \{2[1 + \sqrt{1 + 3x}]^2\}.$$

After dividing by x^2 and taking $x \rightarrow 0$, the limit is $-9/[2(2)^2] = -9/8$.

A2. Derivative from first principles plus interpretation

8 marks

From the limit definition:

$$f'(a) = \lim_{h \rightarrow 0} \{ [1/(a+h+1)] - [1/(a+1)] \} / h.$$

The numerator simplifies to $-h/[(a+h+1)(a+1)]$, so

$$f'(a) = \lim_{h \rightarrow 0} -1/[(a+h+1)(a+1)] = -1/(a+1)^2.$$

At $x = 1$, the point is $(1, 1/2)$ and the slope is $-1/4$. Therefore the tangent line is $y - 1/2 = (-1/4)(x - 1)$, or $y = 3/4 - x/4$.

Since $f''(x) = 2/(x+1)^3 > 0$ near $x = 1$, the curve is concave up, so the tangent line lies below the curve locally.

WORKED SOLUTIONS KEY | SECTION A

High School Calculus - continued

A3. Optimization with modeling**8 marks**

If the upper corners are $(x, 12 - x^2)$ and $(-x, 12 - x^2)$, then the width is $2x$ and the height is $12 - x^2$, where $0 \leq x \leq \sqrt{12}$.

$$A(x) = 2x(12 - x^2) = 24x - 2x^3.$$

$A'(x) = 24 - 6x^2 = 6(4 - x^2)$, so the critical point in the domain is $x = 2$. Also, $A''(2) = -24 < 0$, so $x = 2$ gives a maximum.

Thus the width is 4 and the height is $12 - 4 = 8$. The maximal rectangle has dimensions 4 by 8.

A4. Integral and interpretation**8 marks**

The displacement is

$$\int_0^5 (t^2 - 4t + 3) dt = [t^3/3 - 2t^2 + 3t]_0^5 = 20/3.$$

Factor $v(t) = (t-1)(t-3)$, so the sign changes at $t = 1$ and $t = 3$. With $F(t) = t^3/3 - 2t^2 + 3t$,

$$\begin{aligned} \text{Total distance} &= [F(1)-F(0)] - [F(3)-F(1)] + [F(5)-F(3)] \\ &= 4/3 + 4/3 + 20/3 = 28/3. \end{aligned}$$

They differ because displacement is the signed change in position, while total distance counts all motion as positive.

WORKED SOLUTIONS KEY | SECTION B

Basic Proof Techniques

B1. Direct proof / contradiction hybrid

6 marks

Prove the contrapositive: if n is odd, then n^2 is odd.

Let $n = 2k + 1$ for some integer k . Then

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1,$$

which is odd. Therefore the contrapositive is true, so if n^2 is even, then n is even.

B2. Divisibility proof

6 marks

Factor:

$$n^3 - n = n(n^2 - 1) = n(n-1)(n+1).$$

These are three consecutive integers. Among any three consecutive integers, one is divisible by 3 and at least one is even. Therefore their product is divisible by both 2 and 3, hence divisible by 6.

B3. Sets and logic

4 marks

The statement is true.

If $x \in A \cap (B \cup C)$, then $x \in A$ and $x \in B$ or $x \in C$. Thus $x \in A \cap B$ or $x \in A \cap C$, so $x \in (A \cap B) \cup (A \cap C)$.

Conversely, if $x \in (A \cap B) \cup (A \cap C)$, then either $x \in A \cap B$ or $x \in A \cap C$. In either case $x \in A$ and $x \in B \cup C$. Hence $x \in A \cap (B \cup C)$.

Therefore the two sets are equal by double inclusion.

B4. Counterexample detection

4 marks

For real numbers, the statement is true: if $ab = 0$ and a is nonzero, dividing by a gives $b = 0$. Symmetrically, if b is nonzero then $a = 0$.

For matrices, the statement is false. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then AB is the zero matrix although neither A nor B is the zero matrix. Thus the zero-product property holds in $\mathbb{R}$ but fails for 2×2 matrices.

WORKED SOLUTIONS KEY | SECTION C

Elementary Linear Algebra

C1. System solving with interpretation**6 marks**

Subtract the first equation from the third: $y - z = 1$. Subtract twice the first equation from the second: $y - z = 1$ again.

Thus only two independent equations remain. From $y = z + 1$, the first equation gives $x + z = 1$. Let $z = t$. Then

$$(x, y, z) = (1 - t, 1 + t, t).$$

There is not a unique solution. There are infinitely many solutions forming a line in $\mathbb{R}^3$ because one free parameter remains.

C2. Linear dependence / span**7 marks**

Observe that

$$v_1 + v_2 = (1, 2, 1) + (2, 1, 3) = (3, 3, 4) = v_3.$$

Therefore the set is linearly dependent. One relation is $v_3 = v_1 + v_2$, equivalently $v_1 + v_2 - v_3 = 0$.

Geometrically, all three vectors lie in the span of v_1 and v_2 ; v_3 adds no new direction.

C3. Matrix transformation**6 marks**

$$A(x, y)^T = (2x + y, x + 2y)^T.$$

Solve $2x + y = 3$ and $x + 2y = 3$. Subtracting gives $x - y = 0$, so $x = y$. Then $3x = 3$, hence $x = y = 1$.

There is exactly one solution vector: $(1, 1)^T$. The determinant of A is $4 - 1 = 3$, which is nonzero.

Therefore the map is invertible and cannot collapse the plane to a line.

C4. Eigenvalue readiness lite**6 marks**

The characteristic polynomial is $\det(B - \lambda I) = \lambda^2 - 1$, so the eigenvalues are $\lambda = 1$ and $\lambda = -1$.

For $\lambda = 1$, $Bv = v$ gives $y = x$, so one eigenvector is $(1, 1)$. For $\lambda = -1$, $Bv = -v$ gives $y = -x$, so one eigenvector is $(1, -1)$.

Geometrically, B swaps coordinates: (x, y) maps to (y, x) . It is reflection across the line $y = x$.

WORKED SOLUTIONS KEY | SECTION D

High School Mechanics and E&M

D1. Newtonian modeling on an incline

7 marks

Forces: weight mg downward, normal force N perpendicular to the plane, and kinetic friction $f_k = \mu_k N$ up the plane because the motion is down the plane.

Perpendicular to the plane, $N = mg \cos \theta$. Along the plane, the net force is $mg \sin \theta - \mu_k mg \cos \theta$. Therefore

$$a = g(\sin \theta - \mu_k \cos \theta).$$

For the block to slide downward, $\sin \theta > \mu_k \cos \theta$, equivalently $\tan \theta > \mu_k$.

D2. Work-energy with variable interpretation

6 marks

At maximum height the speed is zero. Conservation of mechanical energy gives

$$(1/2)mv_0^2 = mgh_{\max}, \text{ so } h_{\max} = v_0^2/(2g).$$

Using kinematics, $v^2 = v_0^2 + 2a(y - y_0)$. Setting $v = 0$ and $a = -g$ gives $0 = v_0^2 - 2gh_{\max}$, hence the same result.

The methods agree because both are valid consequences of the same constant-gravity model without air resistance.

D3. Momentum and conservation judgment

6 marks

Initial momentum = $m(u) + 2m(-u/2) = mu - mu = 0$. Since the objects stick together, the total mass is $3m$ and the final momentum is $3mv_f$. Therefore $v_f = 0$.

Initial kinetic energy = $(1/2)mu^2 + (1/2)(2m)(u/2)^2 = 3mu^2/4$. Final kinetic energy is 0.

Kinetic energy is not conserved; the loss is $3mu^2/4$.

D4. Basic E&M field-force reasoning

6 marks

Place $+q$ at $x = 0$ and $+4q$ at $x = d$. Between them the fields oppose, so set magnitudes equal:

$$kq/x^2 = 4kq/(d - x)^2.$$

Thus $(d - x)^2 = 4x^2$. Since $0 < x < d$, $d - x = 2x$, so $x = d/3$. The zero-field point lies one-third of the way from $+q$ toward $+4q$.

The potential cannot be zero between the charges because $V = kq/x + 4kq/(d - x)$, a sum of positive terms. At the zero-field point $E = 0$, so a test charge $+Q$ experiences net force $F = QE = 0$.

WORKED SOLUTIONS KEY | OPTIONAL

Bonus Solutions

X1. Cross-topic maturity question**5 marks**

The statement is false. Take $f(x) = x^3$. Then $f'(x) = 3x^2$, so $f'(0) = 0$, but $x = 0$ is neither a local maximum nor a local minimum because f changes from negative to positive values through 0.

A zero derivative is necessary for a smooth interior extremum, but it is not sufficient.

X2. Mathematical physics bridge question**5 marks**

The statement is false. In $\mathbb{R}^2$, nonzero perpendicular vectors can have dot product zero; for example $u = (1,0)$ and $v = (0,1)$.

Thus $u \cdot v = 0$ does not imply that one vector is zero. This differs from real numbers, where $ab = 0$ implies $a = 0$ or $b = 0$ because $\mathbb{R}$ has the zero-product property.

The dot product is not ordinary multiplication; it can vanish because of orthogonality rather than because one input is zero.

ASSESSMENT GUIDE

Grading Sheet and Course-Fit Rubric

Question-level rubric

Question	Max	Core scoring criteria	Common red flags
A1	6	Correct algebraic strategy, clean rationalization, correct limit.	Cancelling before justified; unsupported guessing.
A2	8	Limit definition, tangent line, concavity interpretation.	Quotient simplification errors; wrong tangent data.
A3	8	Model area correctly, optimize, verify maximum.	Missing domain check; derivative mistakes.
A4	8	Separate displacement from total distance with sign analysis.	Ignoring sign changes at $t = 1$ and $t = 3$.
B1-B4	20	Logical structure, proof validity, correct counterexample use.	Examples used as proofs; scalar and matrix logic conflated.
C1-C4	25	Accurate solving plus interpretation of structure and transformations.	Pure computation with no interpretation; determinant misuse.
D1-D4	25	Model-first physics, conservation conditions, force/field clarity.	Formula dumping; sign errors; force-field-potential confusion.

Readiness bands

Band	Score range	Interpretation
Fully ready	85-100	Ready for rigorous coaching in most offerings; strong candidate for advanced pathways if proof and linear algebra are also strong.
Ready with controlled weaknesses	70-84	Can begin targeted course work, but should be streamed according to section profile.
Borderline	55-69	Needs remediation or a bridge program before full proof-heavy or advanced work.
Not ready	40-54	Foundation repair required before advanced tutoring is likely to be efficient.
Rebuild required	Below 40	Substantial prerequisite reconstruction needed.

ASSESSMENT GUIDE

Course-Specific Cutoffs

Course	Recommended threshold	Section conditions
Calculus I & II Mastery	Overall ≥ 65	Section A $\geq 18/30$ and no catastrophic algebraic weakness.
Linear Algebra Mastery	Overall ≥ 68	Sections B + C $\geq 30/45$.
Ordinary Differential Equations	Overall ≥ 72	A $\geq 20/30$, C $\geq 14/25$, D $\geq 14/25$.
Physics Problem Solving: Mechanics	Overall ≥ 68	D $\geq 17/25$ and A $\geq 16/30$.
Quantum Mechanics: Mathematical Foundations	Overall ≥ 80	A $\geq 22/30$, B $\geq 14/20$, C $\geq 18/25$; X2 strongly preferred.

Suggested use

- Give the written exam first under timed conditions.
- Then conduct a 10-15 minute oral review of one proof, one linear-algebra interpretation, and one mechanics-modeling solution.
- Use the oral review to distinguish memorized procedures from genuine mathematical maturity.

Scoring note: The 100-point readiness score uses Sections A-D. The two bonus questions add up to 10 optional marks and can be used as evidence of mathematical maturity.